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Normalizability of One-dimensional Quasi-exactly Solvable Schrödinger Operators

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Abstract. We completely determine necessary and sufficient conditions for the normalizability of the wave functions giving the algebraic part of the spectrum of a quasi-exactly solvable Schrödinger operator on the line. Methods from classical invariant theory are employed to provide a complete list of canonical forms for normalizable quasi-exactly solvable Hamiltonians and explicit normalizability conditions in general coordinate systems.

1. Introduction

Lie algebraic and Lie group theoretic methods have played a significant role in the development of quantum mechanics since its inception. In the classical applications, the Lie group appears as a symmetry group of the Hamiltonian operator, and the associated representation theory provides an algebraic means for computing the spectrum. Of particular importance are the exactly solvable problems, such as the harmonic oscillator or the hydrogen atom, whose point spectrum can be completely determined using purely algebraic methods. In the early 1980's, in order to study molecular spectroscopy, Alhassid, Gürsey, Iachello, Levine, and collaborators, [2, 3, 1, 14], introduced the concept of a "spectrum generating algebra" to construct models for complicated molecules whose point spectrum could be analyzed algebraically. The Schrödinger operators amenable to the algebraic approach to scattering assumed a "Lie algebraic form," meaning that they belong to the universal enveloping algebra of the spectrum generating algebra. Thus, a second order differential operator

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