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Renormalization Group and the Ginzburg-Landau Equation

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Abstract. We use Renormalization Group methods to prove detailed long time asymptotics for the solutions of the Ginzburg-Landau equations with initial data approaching, as $x \to \pm\infty$, different spiraling stationary solutions. A universal pattern is formed, depending only on this asymptotics at spatial infinity.

1. Introduction

Parabolic PDE's often exhibit universal scaling behavior in long times: the solution behaves as $u(x,t) \sim t^{-\frac{\alpha}{2}} f^*(t^{-\frac{\beta}{2}} x)$ as $t \to \infty$, where the exponents α and β and the function f^* are *universal*, i.e. independent on the initial data and equation, in given classes. This fact has an explanation in terms of the Renormalization Group (RG) [10, 11, 4], very much like the similar phenomenon in statistical mechanics.

In [4] a mathematical theory of this RG was developed and here we would like to apply these ideas to a concrete situation, namely the Ginzburg-Landau equation

$$\dot{u} = \partial^2 u + u - |u|^2 u \,, \tag{1}$$

where $u: \mathbf{R} \times \mathbf{R} \to \mathbf{C}$, is complex, $\partial = \frac{\partial}{\partial x}$ and the dot denotes the time derivative. Equation (1) has a two parameter family of stationary solutions

$$u_{q\theta}(x) = \sqrt{1-q^2} e^{i(qx+\theta)} \,, \tag{2}$$

and a natural question is to inquire about the time development of initial data $u(x)$ which approach two solutions at $\pm\infty$:

$$\lim_{x \to \pm\infty} |u(x) - u_{q_\pm \theta_\pm}(x)| = 0 \,. \tag{3}$$

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