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Quantum Ergodicity on the Sphere

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Abstract. We prove that a random orthonormal basis of eigenfunctions on the standard sphere has quantum ergodic behavior.

As the title portends, this paper is about quantum ergodicity in the most completely integrable of examples: the Laplacian on S^2. The notion of quantum ergodicity we pursue here is the one which characterizes ergodicity of a Schrödinger (or Laplace) operator H in terms of the semi-classical behaviour of its eigenfunctions ([Sn, V, Be, Z.1–2, CdeV.2, HMR, ST]). Roughly, H is quantum ergodic if its orthonormal bases $\{\varphi_j\}$ of eigenfunctions have the following property: $(A\varphi_j, \varphi_j) \to \int_{S^*M} \sigma_A d\mu$ $(j \to \infty)$ for any 0^{th} order ΨDO A ($d\mu$ = Liouville measure, σ_A = principal symbol). This limit formula is a kind of quantum analogue of the Birkhoff ergodic theorem and is known to hold whenever the classical (e.g. geodesic) flow is ergodic. Otherwise it does not seem well understood: for example, it might (for all that is proved to date) even hold for a generic Laplacian. Our purpose here is, perversely, to investigate it on the sphere. Of course, the usual basis $\{Y^l_m\}$ of spherical harmonics does not have the ergodic property. But, due to the high degeneracy of eigenvalues, there is an infinite dimensional manifold of orthonormal bases of eigenfunctions. This manifold is actually a group and carries a unit mass Haar measure. Our main result is that, relative to this measure, almost all bases have the ergodic property.

To state the result more precisely we will need to introduce some terminology and background.

Throughout this paper we will be considering only the standard 2-sphere S^2, although our methods would work on many other spaces. We will usually omit explicit reference to the metric on S^2; all notation such as $L^2(S^2)$, Δ, etc., will refer to the standard metric.

We first recall that

$$L^2(S^2) = \bigoplus_l E_l \quad (\dim E_l = 2l + 1), \tag{1.1}$$

where E_l is the complex eigenspace of spherical harmonics of degree l. Equivalently, E_l is the eigenspace of the laplacian Δ of eigenvalue $l(l + 1)$. We will let π_l denote