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Real Polarization of the Moduli Space of Flat Connections on a Riemann Surface*

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Abstract. We prove that the moduli space of flat $SU(2)$ connections on a Riemann surface has a real polarization, that is, a foliation by Lagrangian subvarieties. This polarization may provide an alternative quantization of the Chern–Simons gauge theory in higher genus, in line with the results of [11] for genus one.

I. Introduction

Let Σ^g be a 2-manifold of genus g. The space $\bar{\mathscr{S}}_g$ of conjugacy classes of representations $\rho:\pi_1(\Sigma^g)\to G$, where G is a compact lie group, is an algebraic variety containing an open set $\mathscr{S}_g$ which is a symplectic manifold. The symplectic form ω is the Chern class of a line bundle $\mathscr{L}\to\mathscr{S}_g$ which extends to a line bundle $\bar{\mathscr{L}}\to\bar{\mathscr{S}}_g$. The line bundle $\bar{\mathscr{L}}\to\bar{\mathscr{S}}_g$ is endowed with a canonical connection and hermitian metric. Furthermore, a choice of a metric on Σ^g endows $\mathscr{S}_g$ with a complex structure making the symplectic form ω Kähler and the line bundle $\mathscr{L}$ holomorphic. Thus, ignoring for a moment the singularities of $\bar{\mathscr{S}}_g$, we have arrived at the natural setting for quantization; namely, we have been given a symplectic manifold $\mathscr{S}_g$, a line bundle $\mathscr{L}\to\mathscr{S}_g$ with connection of curvature ω, and a polarization of the sheaf of local sections of $\mathscr{L}\to\mathscr{S}_g$.

Recent developments have emphasized the importance of this system in relation to the theory of representations of loop groups, conformal field theory, and 3-dimensional topological quantum field theory. For example, the quantization of the above system in $g=1$ can be naturally associated to the Weyl–Kac characters of the integrable representations of the Kac–Moody lie algebra $\hat{G}$ associated to G; while this quantization for general g yields a projectively flat bundle over moduli space associated to the conformal field theory of G current algebra.

The main motivation for our study of this system is however related to Chern–Simons gauge theory and the topological field theory related to it [12].

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