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Growth and Integrability in the Dynamics of Mappings

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Abstract. The growth of some numerical characteristics of the mappings under their iterations in the context of the general problem of integrability is discussed. In the general case such characteristics as complexity by Arnold or the number of the different images for the multiple-valued mappings are growing exponentially. It is shown that the integrability is closely related with the *polynomial* growth. The analogies with quantum integrable systems are discussed.

The goal of this paper is to discuss the growth of some concrete numerical characteristics of the mapping under its iterations in the context of the general problem of integrability for such discrete systems (see e.g. [1]). The results can be summarized in a quite natural way: The *integrability* has an essential correlation with the *weak growth* of certain characteristics. One of them is the *complexity* introduced and investigated by Arnold in the recent papers [2]. In the simplest case for the mappings f of the plane the complexity can be defined as the number of intersection points of the fixed curve Γ_1 with the image of the second curve Γ_2 under the k^{th} iteration of f

$$A^f_{\Gamma_1,\Gamma_2}(k) = \#\Gamma_1 \cap f^{(k)}(\Gamma_2)\,.$$

If the mapping f is the polynomial one and the curves Γ_1 and Γ_2 are algebraic, then it is easy to see that the growth of $A^f_{\Gamma_1,\Gamma_2}(k)$ will in general exponential on k, what is in a good agreement with the general Arnold's result [2]. In the first paragraph we will show that for the integrable (in the various senses) polynomial automorphisms of the plane this growth is much weaker

$$A^f_{\Gamma_1,\Gamma_2}(k) < C(\Gamma_1,\Gamma_2,f)\,,$$

and this property turns out to be characteristic for the integrable mappings in this case. In the rest of the paper we discuss the dynamics of the multiple-valued mappings (correspondences) Φ and the growth of the numbers $N^\Phi_x(k)$ of the different images