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Exponent Inequalities for the Bulk Conductivity of a Hierarchical Model

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Abstract. The bulk conductivity $\sigma^*(p)$ of the bond lattice in $\mathbb{Z}^d$ is considered, where the bonds have conductivity 1 with probability p or $\varepsilon \geqq 0$ with probability $1-p$. Various representations of the derivatives of $\sigma^*(p)$ are developed. These representations are used to analyze the behavior of $\sigma^*(p)$ for $\varepsilon=0$ near the percolation threshold p_c, when the conducting backbone is assumed to have a hierarchical node-link-blob (NLB) structure. This model has loops on arbitrarily many length scales and contains both singly and multiply connected bonds. Exact asymptotics of $\frac{d^2\sigma^*}{dp^2}$ for the NLB model are proven under some technical assumptions. The proof employs a novel technique whereby $\frac{d^2\sigma^*}{dp^2}$ for the NLB model with $\varepsilon=0$ and p near p_c is computed using perturbation theory for $\sigma^*(p)$ (for two- and three-component resistor lattices) around $p=1$ with a sequence of ε's converging to 1 as one goes deeper in the hierarchy. These asymptotics establish convexity of $\sigma^*(p)$ (for the NLB model) near p_c, and that its critical exponent t obeys the inequalities $1 \leqq t \leqq 2$ for $d=2, 3$, while $2 \leqq t \leqq 3$ for $d \geqq 4$. The upper bound $t=2$ in $d=3$, which is realizable in the NLB class, virtually coincides with two very recent numerical estimates obtained from simulation and series expansion for the original model.

1. Introduction

The transport properties of disordered media play an important role in many branches of science and engineering. For example, disordered conductors arise naturally in fields as varied as biology, geology, and solid state physics, yet also serve as key building blocks of advanced composite materials. Of particular

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