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Universal Exchange Algebra for Bloch Waves and Liouville Theory

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Abstract. We derive a universal formula for the exchange algebra in the Bloch wave basis. The main tool we use is a lattice version of the Coulomb gas picture of conformal field theory, making its quantum group structure explicit from the very beginning. Calulations are then reduced to a factorization problem in $\mathscr{U}_q(sl_2)$.

1. Introduction

There is an intimate connection between Liouville theory, conformal field theory and quantum groups. As a unifying feature, we will take the Schroedinger equation

$$(\partial_z^2 - \mathscr{U})\,\xi = 0. \tag{1}$$

The relation to Liouville's equation already appeared in Poincaré [1] and is as follows. Let ξ^1 and ξ^2 be two linearly independent solutions of Eq. (1) with their Wronskian normalized to one. Setting

$$e^{-\frac{1}{2}\varphi} = \xi^1 \bar{\xi}^2 - \xi^2 \bar{\xi}^1,$$

it is straightforward to check that the field φ satisfies Liouville's equation

$$\partial_z \partial_{\bar{z}} \varphi = 2e^{\varphi}.$$

The meaning of Eq. (1) in conformal field theory may be easily understood by looking at the transformation properties of this equation under conformal transformations. Let us change coordinates $z \to z(w)$. If ξ behaves like a differential of weight $-\frac{1}{2}$ and $\mathscr{U}$ as a Schwarzian connection, i.e.

$$\xi(w)\,dw^{-\frac{1}{2}} = \xi(z)\,dz^{-\frac{1}{2}},$$
$$\mathscr{U}(w)\,dw^2 = \mathscr{U}(z)\,dz^2 + \tfrac{1}{2}\{z, w\}\,dz^2,$$