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Universality in Orbit Spaces of Compact Linear Groups*

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Abstract. If $\{p^1(x), \ldots, p^q(x)\}$ is a minimal integrity basis of the ideal of polynomial invariants of a compact coregular linear group G, the orbit map

$$p = (p^1(x), \ldots, p^q(x)) : \mathbb{R}^n \to \mathbb{R}^q,$$

yields a diffeomorphic image $\bar{\mathcal{P}} = p(\mathbb{R}^n) \subseteqq \mathbb{R}^q$ of the orbit space $\mathbb{R}^n/G$. Starting from this fact, we point out some properties which are common to the orbit spaces of all the compact coregular linear groups of transformations of $\mathbb{R}^n$. In particular we show that a contravariant metric matrix $\hat{P}(p)$ can be defined in the interior of $\bar{\mathcal{P}}$, as a polynomial function of $(p^1, \ldots, p^q)$. We prove that the matrix $\hat{P}(p)$, which characterizes the set $\bar{\mathcal{P}}$, as it is positive semi-definite only for $p \in \bar{\mathcal{P}}$, can be determined as a solution of a canonical differential equation, which, for every compact coregular linear group, depends only on the number q and on the degrees of the elements of the minimal integrity bases. This allows to determine all the isomorphism classes of the orbit spaces of the compact coregular linear groups through a determination of the equivalence classes of the corresponding matrices $\hat{P}(p)$. For $q \leqq 3$ (orbit spaces with dimensions $\leqq 3$), the solutions $\hat{P}(p)$ of the canonical equation are explicitly determined and the number of their equivalence classes is shown to be finite. It is also shown that, with a convenient choice of the minimal integrity basis, the polynomial matrix elements of $\hat{P}(p)$ have only integer coefficients. Arguments are given in favour of the conjecture that our conclusions hold true for all values of q. Our results are relevant and lead to universality properties in the physics of spontaneous symmetry breaking.

1. Introduction

In theories in which the ground state of the system is determined by a stationary point of a potential which is invariant under the transformations of a compact

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