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On The Origin of Integrability in Matrix Models

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Abstract. The matrix integrals involved in $2d$ lattice gravity are studied at finite N. The integrable systems that arise in the continuum theory are shown to result directly from the formulation of the matrix integrals in terms of orthogonal polynomials. The partition function proves to be a tau function of the Toda lattice hierarchy. The associated linear problem is equivalent to finding the polynomial basis which diagonalizes the partition function. The cases of one Hermitian matrix, one unitary matrix, and Hermitian matrix chains all fall within the Toda framework.

1. Introduction

Recently a great deal of progress has been made in the nonperturbative formulation of low-dimensional toy models of string theory – two-dimensional gravity coupled to $c < 1$ matter – by formulating the theory in terms of large-N matrix integrals [1, 2]. Soluble matrix integrals give the full partition function for string theory in simple backgrounds. One of the most remarkable features is that the continuum theory is governed by the KP hierarchy of commuting differential operators [2–4]. The Lax operators of KP provide different realizations of the Heisenberg algebra of the spectral parameter and its conjugate momentum [4], and the commuting KP flows parametrize the space of gravitational field theories for low c_{matter}. Continuum analyses suggest [3, 5] that the partition function is a tau function of the KP hierarchy.

These beautiful results raise many issues. One is to understand better the origin of the integrable systems that appear at the level of the matrix integrals. Exactly what is the connection of integrability to $2d$ gravity? Are there gravity theories for each integrable system having a thermodynamic limit? What is the precise nature of the tau function that appears? We would also like to connect the matrix model formulation to other approaches to $2d$ gravity and string theory. It is important to obtain lattice expressions for scaling operators in order to compare with

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