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On the Complete Integrability of Completely Integrable Systems

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Abstract. The question of complete integrability of evolution equations associated to $n \times n$ first order isospectral operators is investigated using the inverse scattering method. It is shown that for $n > 2$, e.g. for the three-wave interaction, additional (nonlinear) pointwise flows are necessary for the assertion of complete integrability. Their existence is demonstrated by constructing action-angle variables. This construction depends on the analysis of a natural 2-form and symplectic foliation for the groups $GL(n)$ and $SU(n)$.

1. Introduction

A classical Hamiltonian flow with $2N$ degrees of freedom is said to be completely integrable if it has N independent integrals of the motion which are in involution. More generally, k independent commuting Hamiltonian flows in a $2N$-dimensional manifold are said to be a completely integrable family if there are $N-k$ independent integrals of the motions which are in involution, or equivalently the $N-k$ flows may be enlarged to a set of N independent commuting flows. By a theorem of Jacobi and Liouville, there then exist (at least locally in phase space) a new set of canonical variables, called action-angle variables, in which the flows are particularly simple; see [A] for a precise global version due to Arnold.

In recent years a number of nonlinear evolution equations, beginning with the KdV equation, have been shown to have Hamiltonian form on appropriate infinite-dimensional manifolds and to have an infinite family of integrals of the motion which are in involution. Such equations are commonly referred to as "completely integrable," although it no longer makes sense to count half the number of dimensions. Nevertheless the inverse scattering method makes it possible to give a precise form to the question of complete integrability and, indeed, to reduce it to a question in a finite dimensional space.

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