Commun. Math. Phys. 138, 33–49 (1991)

Communications in
Mathematical
Physics

Curve-Straightening in Closed Euclidean Submanifolds

Anders Linnér

Department of Mathematics, University of Missouri-St. Louis, St. Louis, Missouri 63121-4499, USA

Received March 10, 1989; in revised form November 5, 1990

Abstract. The flow in the negative direction of the gradient vector field associated with the functional total squared (geodesic) curvature $\int k^2 ds$ is the so-called curve-straightening flow. This paper will consider spaces of closed curves in closed Euclidean submanifolds. It will define these spaces of curves as submanifolds of certain Hilbert manifolds representing all curves. The main result will then be to show the existence of a particular set of functionals defined on the entire Hilbert manifold which have the following four properties: 1. The directional derivatives of these functionals may be computed by solving an initial value problem for a system of ordinary differential equations. 2. By introducing a suitable Hilbert space basis for the Sobolev spaces used, the gradients may be effectively computed (but of course not explicitly computed, except in very special cases). 3. The gradients span the space normal to the tangent space of the space of closed curves. 4. Despite the fact that these gradients in general are not given explicitly it is nevertheless possible to compute the projection onto the tangent space to the space of closed curves. In particular we do this for the gradient of $\int k^2 ds$. When all details are worked out this gives us an algorithm (which we supply) for finding critical points in the space of closed curves. It is not known if the trajectories actually always converge to critical points. If the functional is modified to include a multiple of the length so the functional becomes $\int k^2 + \lambda ds$ then the above convergence is known for $\lambda > 0$. The motivating application for the curve-straightening flow is the possibility of using it to find (non-trivial) closed (periodic) geodesics. Note that if $\lambda = 0$ then a closed geodesic is a global minimum. For any λ, geodesics are critical but there are also other critical points, the so-called elastic curves. The paper concludes by deriving the second variation formula for $\int k^2 + \lambda ds$ along closed geodesics. The quadratic functional associated with the second derivative is shown to be positive definite even for non-zero λ along some closed geodesics in some particular manifolds of interest.