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Matrix Integration and Combinatorics of Modular Groups

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Abstract. Integration over Gaussian matrix ensembles was used to obtain the virtual Euler characteristics of mapping class groups. We present some simplifications in the combinatorial part of the calculation.

1. Introduction

In a beautiful work, Harer and Zagier [1] obtained the (virtual) Euler characteristics of the mapping class (or modular) group of punctured orientable compact Riemann surfaces of genus g in two steps. The first of topological nature is the reduction to a purely combinatorial problem. The second one is the solution of this combinatorial problem using a variety of techniques, including at some stage an integration over a Gaussian ensemble of random matrices. This point was further elaborated by Penner [2] who used a dual version of this integration. Penner made contact with physicists' work initiated by 't Hooft [3] and further developed in [4–7] on the topological interpretation of the large N expansion of (field theoretic) models. In this note we shall not attempt to describe the topological background, referring the reader to the expositions of the previously mentioned authors [1, 2] as well as the review by Ivanov [8]. Rather we would like to show that the combinatorial calculations in both references [1] and [2] can be substantially reduced if one appeals to fairly standard tools. We shall present three elementary calculations. The first is purely group theoretical, and relies on the Frobenius duality formula between the linear and symmetric groups. In essence, it had already been sketched in the last appendix of [6] and was based on a suggestion of J.-M. Drouffe. Parenthetically, we obtain a strikingly simple expression for the average of an irreducible polynomial character over the Gaussian Hermitian ensemble [formula (2.13) below], which awaits presumably adequate generalizations.

A second calculation is even simpler and only requires an elementary knowledge of the harmonic oscillator. It is candidate for generalizations based on an interpretation in terms of free fermions, too.

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