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An Application of Aomoto–Gelfand Hypergeometric Functions to the $SU(n)$ Knizhnik–Zamolodchikov Equation

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Abstract. Solutions to the Knizhnik–Zamolodchikov equation for Verma modules of the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ are explicitly given by certain integrals called Aomoto–Gelfand hypergeometric functions.

1. Introduction

The starting point of our study was the result of Christe and Flume [4], which gave explicit integral representations of the 4-point functions of the $SU(2)$ Wess–Zumino–Witten model as solutions to the Knizhnik–Zamolodchikov equation. Similar results were previously obtained by Zamolodchikov and Fateev [13]. On the other hand, Aomoto [1], [2] studied the integrals of the following kind and derived a system of differential equations for them with respect to variables $z_1, \ldots, z_N$:

$$\int \Phi \varphi \, dt_1 \cdots dt_m,$$
$$\Phi = \prod_{i,a} (t_i - z_a)^{\lambda_{ai}} \prod_{i,j} (t_i - t_j)^{\nu_{ij}} \prod_{a,b} (z_a - z_b)^{\mu_{ab}}. \tag{1.1}$$

Here φ are rational functions whose poles are contained in the diagonal set $\bigcup_{i,a} \{t_i = z_a\} \cup \bigcup_{i,j} \{t_i = t_j\} \cup \bigcup_{a,b} \{z_a = z_b\}$, and $\lambda_{ai}, \nu_{ij}, \mu_{ab}$ are complex parameters. This kind of integrals are generalizations of hypergeometric function, and Gelfand and others studied a class of generalized hypergeometric functions including (1.1) ([12]). We call them 'Aomoto–Gelfand hypergeometric functions'.

If the parameters $\lambda_{ai}, \nu_{ij}, \mu_{ab}$ take certain values, then the integral (1.1) reduces to the one of Christe and Flume. In this case, Aomoto's differential equation is nothing but the Knizhnik–Zamolodchikov equation. A similar result on the n-point functions was obtained by Date et al. [6].

In this paper, we shall generalize the last result to the $SU(n)$ Knizhnik–Zamolodchikov equation. We briefly sketch our construction.