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Entropy of Bogoliubov Automorphisms of the Canonical Anticommutation Relations

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Abstract. We compute the entropy $h_{\omega_A}(\alpha_U)$ in the sense of Connes, Narnhofer and Thirring of Bogoliubov automorphisms α_U of the CAR-algebra with respect to invariant quasifree states ω_A with $0 \leqq A \leqq 1$ having pure point spectrum.

1. Introduction

In their recent paper [3] Connes, Narnhofer, and Thirring extended the definition of entropy for automorphisms of finite von Neumann algebras studied in [4] to the case of automorphisms of C^*-algebras invariant with respect to a given state. In the present paper we shall compute this for Bogoliubov automorphisms of the CAR-algebra with respect to invariant quasifree states. Recall, for more details see Sect. 4, that if H is a complex Hilbert space and $f \to a(f)$ is a representation of H in the CAR-algebra $\mathscr{A}(H)$ satisfying the canonical anticommutation relations then each unitary operator U on H defines a Bogoliubov automorphism α_U of $\mathscr{A}(H)$ by $\alpha_U(a(f)) = a(Uf)$. If $A \in B(H)$ satisfies $0 \leqq A \leqq 1$ and $AU = UA$, then α_U is invariant with respect to the (gauge invariant) quasifree state ω_A defined by A. In the case $A = \frac{1}{2}1$, i.e. ω_A is the unique tracial state τ on $\mathscr{A}(H)$, then the entropy $h_\tau(\alpha_U)$ is the same as that of the extension of α_U to the GNS-representation of τ as defined in [4]. A. Connes suggested to us that the formula for the entropy should be

$$h_\tau(\alpha_U) = \frac{\log 2}{2\pi} \int_0^{2\pi} m(U)(\theta) d\theta, \tag{1}$$

where $m(U)$ is the multiplicity function of the absolutely continuous part U_a of U, a conjecture which initiated the present work. More generally, if U_a acts on the

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