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The Time Dependent Amplitude Equation for the Swift-Hohenberg Problem

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Dedicated to Res Jost and Arthur Wightman

Abstract. Precise estimates for the validity of the amplitude approximation for the Swift-Hohenberg equation are given, in a fully time dependent framework. It is shown that small solutions of order $\mathcal{O}(\varepsilon)$ which are modulated like stationary solutions have an evolution which is well described in the amplitude approximation for a time of order $\mathcal{O}(\varepsilon^{-2})$. For the proofs, we use techniques for nonlinear semigroups and oscillatory integrals.

1. Introduction

In this paper, we study the relation between a multi-scale nonlinear problem and its associated amplitude equation in a fully time-dependent framework. In order to keep the exposition sufficiently simple, we state and prove our results in the framework of the Swift-Hohenberg equation

$$\partial_t u(x,t) = (3\varepsilon^2 - (1+\partial_x^2)^2)u(x,t) - u^3(x,t). \tag{1.1}$$

Here, u is a function $\mathbf{R}\times\mathbf{R}^+ \to \mathbf{R}$. This equation has been studied in detail in [1], and we summarize those of the results which are relevant for the current study.

1. Equation (1.1) has stationary (i.e., time-independent) solutions, for small ε which are of the form

$$u(x,t) \approx 2\varepsilon\cos(x). \tag{1.2}$$

2. Equation (1.1) has front solutions which are of the form

$$u(x,t) = \varepsilon \sum_{n\in\mathbf{Z}} u_n(\varepsilon x - \varepsilon^2 ct)e^{inx},$$

with the reality conditions

$$u_n(y) = \bar{u}_{-n}(y).$$