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The Trotter-Kato Product Formula for Gibbs Semigroups

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Abstract. The trace-norm convergence of the Trotter-Lie product formula has recently been proved for particular classes of Gibbs semigroups. In the present paper we prove it for the whole generality including generalization of the product formula proposed by Kato.

1. Introduction

Let A and B be linear operators in a separable complex Hilbert space $\mathscr{H}$. Then, under suitable conditions concerning $\{A, B\}$ the strong limit

$$s\text{-}\lim_{n\to\infty}\left(\exp\left(-\frac{t}{n}A\right)\exp\left(-\frac{t}{n}B\right)\right)^n=\exp(-tC) \tag{1.1}$$

exists for $t \geqq 0$, where the operator C can be constructed by means of A and B. This is the well-known Trotter-Lie product formula for strongly continuous (C_0-) semigroups [1]. (For finite matrices it has been established by Sophus Lie about 1875.) Since the discovery of the product formula, it has permeated through mathematics and mathematical physics, challenging the problem of relaxation and generalization of the hypotheses under which the formula holds, see [2–10].

A solution of this problem implies that one has to do the following:

(i) to find the set of pairs $\{A, B\}$ for which the limit (1.1) exists;
(ii) to identify the operator C and to describe the mapping $\{A, B\}: \to C$;
(iii) to generalize (if possible) the exponential functions involved in the left-hand side of (1.1) to a class of real-valued, Borel measurable functions $f(\cdot)$, $g(\cdot)$ such that in some operator topology τ,

$$\tau\text{-}\lim_{n\to\infty}\left(f\left(\frac{t}{n}A\right)g\left(\frac{t}{n}B\right)\right)^n=\exp(-tC)\Pi \tag{1.2}$$

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