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Isoholonomic Problems and Some Applications

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Abstract. We study the problem of finding the shortest loops with a given holonomy. We show that the solutions are the trajectories of particles in Yang-Mills potentials (Theorem 4), or, equivalently, the projections of Kaluza-Klein geodesics (Theorem 2). Applications to quantum mechanics (Berry's phase, Sect. 3) and the optimal control of deformable bodies (Sect. 6) are touched upon.

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1. The Problem and an Introduction

1.1 The Problem which we investigate is the isoholonomic problem: among all loops with a fixed holonomy, find the loop of minimum length.

The data needed to formulate this problem are a principal bundle

$$\pi: Q \to X \tag{[1.1]}$$

with connection A, a Riemannian metric k on X, and a point $x_0 \in X$ at which the loop and its holonomy are based. (The holonomy is called the Wilson loop integral, or the path-ordered exponential of -A in the physics literature.) The structure group of the bundle will be denoted by G. It is a Lie group which acts on Q on the right, and such that $X \cong Q/G$.