Commun. Math. Phys. 128, 427–435 (1990)

Communications in Mathematical Physics

Bounds on the Unstable Eigenvalue for Period Doubling

J.-P. Eckmann[1] and H. Epstein[2]

[1] Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland
[2] CNRS and IHES, Bures sur Yvette, France

Abstract. Bounds are given for the unstable eigenvalue of the period-doubling operator for unimodal maps of the interval. These bounds hold for all types of behaviour $|x|^r$ of the interval map near its critical point. They are obtained by finding cones in function space which are invariant under the tangent map to the doubling operator at its fixed point.

1. Introduction

One-parameter families of maps of the interval, such as

$$\mu \mapsto 1-\mu|x|^r ,$$

with $r>1$ fixed, exhibit sequences of bifurcation points for period doubling, which accumulate at a universal rate δ_r, [*F*, *CT*]. For example, for families of quadratic maps (i.e., for $r=2$), it is well known that $\delta_r=4.66920\ldots$. A rigorous bound on δ_2 is given in [EW 2]. The aim of this paper is to give rigorous bounds on δ_r for all $r>1$ by using a convenient reformulation of the tangent map to the doubling operator.

To state these bounds, we need some notation. Consider the Cvitanović-Feigenbaum equation

$$g(x)=-\frac{1}{\lambda}g(g(-\lambda x)),$$

with the conditions $g(0)=1$ and $\lambda=-g(1)$. We consider this equation on several classes of functions, labelled by r. Namely, we require g to be of the form $g(x)=f(|x|^r)$ with f analytic in a neighborhood of $[0,1]$. A solution of this equation is then denoted by g_r. Solutions having some additional properties (we shall spell them out in the next section) are known to exist for all $r>1$. In this paper, we show that these additional properties have the following consequence: Denote $f_r(z)=g_r(z^r)$ for $z\in[0,1]$, and $\lambda_r=-g_r(1)$. Then, δ_r satisfies the inequality

$$\frac{1}{\lambda_r^r}\left(1-\frac{1}{r|f_r'(0)|}\right)<\delta_r<\frac{1}{\lambda_r^r}.$$