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Loop Groups and Yang-Mills Theory in Dimension Two

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Abstract. Given a connection ω in a G-bundle over S^2, then a process called radial trivialization from the poles gives a unique clutching function, i.e., an element γ of the loop group ΩG. Up to gauge equivalence, ω is completely determined by γ and a map $f: S^2 \to \mathfrak{g}$ into the Lie algebra. Moreover, the Yang-Mills functional of ω is the sum of the energy of γ and the square of a certain norm of f. In particular, the Yang-Mills functional has the same Morse theory as the energy functional on ΩG. There is a similar description of connections in a G-bundle over an arbitrary Riemann surface, but so far not of the Yang-Mills functional.

1. Introduction

The purpose of this paper is to point out that the Yang-Mills functional on S^2 with gauge group G essentially is the same as the energy functional on the loop group ΩG. More precisely, I prove that up to gauge equivalence, a connection in a G-bundle P over S^2 is completely described by a loop $\gamma \in \Omega G$ and a map $f: S^2 \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Furthermore, the Yang-Mills functional $\mathcal{YM}(\omega)$ is the sum $\pi\mathcal{E}(\gamma) + \langle f, f \rangle$, of the energy of γ and a certain inner product of f with itself (Theorem 2.6).

In [1] Atiyah and Bott mentioned that the Yang-Mills functional on S^2 and the energy functional on ΩG have the same Morse theory and in [2] Friedrich and Habermann prove it in all details, but it is of course a trivial consequence of the above description of the Yang-Mills functional.

One consequence is that the space of solutions to the Yang-Mills equations on S^2 is the same as the space of homomorphisms $S^1 \to G$. There are similar results in higher dimensions: the space of instantons on $\mathbb{R}^4$ is the same as the space of holomorphic maps $S^2 \to \Omega G$ and the space of monopoles on $\mathbb{R}^3$ is the same as the space of holomorphic maps $S^2 \to G/T$, where T is a maximal torus. In both cases there are suitable boundary conditions at infinity. But the proofs of these results are done in a very roundabout way and the connection between Yang-Mills theory and mapping spaces are not fully understood. The Yang-Mills equation on the