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Quadratic Maps without Asymptotic Measure

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Abstract. An interval map is said to have an asymptotic measure if the time averages of the iterates of Lebesgue measure converge weakly. We construct quadratic maps which have no asymptotic measure. We also find examples of quadratic maps which have an asymptotic measure with very unexpected properties, e.g. a map with the point mass on an unstable fix point as asymptotic measure. The key to our construction is a new characterization of kneading sequences.

1. Introduction

A probability distribution ν on the phase space X of a discrete-time dynamical system $f:X\to X$ is called an asymptotic measure, if the normalized uniform measure λ on the phase space, e.g. Lebesgue measure or more generally a Riemannian volume, tends under the action of the dynamical system to the distribution ν. In mathematical terms this means that $\frac{1}{n}\sum_{k=0}^{n-1}(f^*)^k\lambda$ converges weakly to ν, where f^* is defined by $\int \psi d(f^*\lambda)=\int \psi\circ f d\lambda$ for $\psi\in C(X)$. For many hyperbolic systems asymptotic measures exist, e.g. for axiom-A systems (cf. [B]). Sometimes they are called natural measures or Bowen-Ruelle-Sinai measures.

For nonhyperbolic systems the situation is more complicated. Consider the family $f_a(x)=ax(1-x)$ with $0<a\leqq 4$ of quadratic maps on $[0,1]$. Each f_a has either sensitive dependence to initial conditions $\left(\text{i.e. there is an } \varepsilon>0 \text{ such that } \sup_{n>0} \text{length } (f^nJ)>\varepsilon \text{ for all intervals } J\subseteqq[0,1]\right)$, or there is an attractor (a stable periodic orbit or a Cantor set) which attracts Lebesgue – a.e. trajectory (cf. [G]). In the latter case, the attractor supports a unique f_a-invariant probability measure, which is an asymptotic measure of entropy zero (cf. [Ni, P]).

Work of Jakobson [Ja], Collet/Eckmann [CE 2] and others [Mi, BC, R, No, K 1, K 2, NvS] suggests that for "most" f_a with sensitive dependence there is a unique absolutely continuous invariant probability measure, which, by the