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The Trace Formula for Schrödinger Operators on the Line

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Abstract. This paper discusses certain aspects of the spectral and inverse spectral problems for the Schrödinger operator $L(q) = -\frac{d^2}{dx^2} + q(x)$, for $q(x) \in C(\mathbb{R})$, the space of bounded continuous functions. The trace formula of the title is the relation

$$q(0) = \lambda_0 + \sum_{j=1}^{\infty} (\lambda_{2j} + \lambda_{2j-1} - 2\mu_j)$$

with appropriate choices of $\{\lambda_j\}_{j=0}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$, which is a familiar relation in the theory of Hill's equation. We characterize the set $\mathscr{T} \subseteq C(\mathbb{R})$ of potentials for which this holds. Further extensions of the theory of Hill's equation are also obtained. From the spectrum $\sigma(L(q))$ a torus $T(q)$ is constructed, which is in general infinite dimensional; every $q(x) \in \mathscr{T}$ can be mapped to a continuous path on $T(q)$, described by the auxiliary spectrum $\{\mu_j\}_{j=1}^{\infty}$. Under certain geometrical conditions on $\sigma(L(q))$ this path is the orbit of a C^1 vector field on $T(q)$, and the mapping extends to one from the hull $\mathscr{H}(q) = \overline{\{q(x+\xi); \xi \in \mathbb{R}\}}$ to the closure of this orbit. In particular $\mathscr{H}(q)$ is compact. These results have applications in the theory of Schrödinger operators with ergodic potentials.

1. Introduction

This paper is concerned with the study of the spectral problem for the Schrödinger operator

$$\left(-\frac{d^2}{dx^2} + q(x)\right)\psi = \lambda\psi, \tag{1.1}$$

considered on the line $-\infty < x < +\infty$. We consider potentials $q(x) \in C(\mathbb{R})$, the space of bounded continuous functions. Two standard spectral problems are that

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