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Trace Class Perturbations and the Absence of Absolutely Continuous Spectra

Barry Simon[1] and Thomas Spencer[2]

[1] Division of Physics, Mathematics, and Astronomy, California Institute of Technology, 253-37, Pasadena, CA 91125, USA; Research partially funded under NSF grant number DMS-8801918
[2] Institute for Advanced Study, Princeton, NJ 80309, USA

Dedicated to Roland Dobrushin

Abstract. We show that various Hamiltonians and Jacobi matrices have no absolutely continuous spectrum by showing that under a trace class perturbation they become a direct sum of finite matrices.

1. Introduction

One of the most versatile tools in the study of scattering theory is the trace class theory which goes back to the basic work of Kato, Kuroda, Rosenblum and Birman, and which was raised to a high art by Pearson. A summary of the basic results can be found in Reed-Simon [13].

We will apply these ideas to the study of stochastic Schrödinger operators and Jacobi matrices to show that, typically, there is no absolutely continuous spectrum (at least in one dimension). At first sight, this seems an unlikely tool since there are no scattering states if σ_{ac} is empty. The point is that the trace class theory is ideal for showing that two operators have the same absolutely continuous spectrum so if we can show that under some kind of trace class perturbation h (or its equivalent) can be transformed to an operator without any absolutely continuous spectrum, we are done. In a different context, this idea has recently been used by Howland [4, 5]. Obviously direct sums of finite matrices have no absolutely continuous spectrum and it is these operators which we will show to be equivalent to the original ones.

A simple example concerns one dimensional Jacobi matrices of the form

$$(hu)(n)=u(n+1)+u(n-1)+v(n)u(n) \tag{1}$$

on $l^2(\mathbb{Z})$. If $v(n)=\lambda\cos(\pi\alpha n)$, then for small λ and suitable α, h has absolutely continuous spectrum [1] but if $v(n)=\lambda\tan(\pi\alpha n)$, there is not absolutely continuous spectrum for any $\lambda\neq 0$ if α is irrational [16]. There have been spectulations that this is due to the fact that tan is unbounded and we will prove that this is so in Sect. 2. So