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On the Microscopic Validity of the Wulff Construction and of the Generalized Young Equation

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Abstract. For a large class of $1 + 1$ dimensional interfaces of the Solid-On-Solid type we prove on a microscopic basis the validity of the Wulff construction and of the generalized Young equation which gives the contact angle of a sessile drop on a wall. Our proof relies on a new method to treat random walks with a finite number of global constraints.

1. Introduction

Consider a phase A in a container, whose walls are partially wet by droplets of a phase B. Although small, these droplets are macroscopic, and their contact angle Θ with the wall can be measured and studied as a function of temperature, concentration or any other parameters. A transition from partial wetting to complete wetting may occur, if the angle Θ decreases down to zero, where a thin film of the phase B separates the phase A from the wall.

It is well known that the contact angle Θ is related to the surface tension σ_{AB} and wall free energies σ_{AW}, σ_{BW} through Young's equation (Young 1805). For isotropic media, it reads

$$\sigma_{AB} \cos \Theta = \sigma_{AW} - \sigma_{BW}. \tag{1}$$

The study of droplets and wetting films is also important in metals and other anisotropic media. There Young's equation has to be modified. It takes the form [1]

$$\sigma_{AB}(\Theta, \varphi) \cos \Theta - \sin \Theta \frac{\partial}{\partial \Theta} \sigma_{AB}(\Theta, \varphi) = \sigma_{AW} - \sigma_{BW}. \tag{2}$$

Equation (2) is to be understood as follows: take a point anywhere on the borderline of the droplet. This corresponds to a choice of a direction φ in the plane of the wall. The contact angle $\Theta = \Theta(\varphi)$ is then the angle of the wall with the tangent plane to the droplet at the given point. The function $\sigma_{AB}(\Theta, \varphi)$ is the A–B interfacial free energy per unit area of a flat A–B interface which would be parallel to the given tangent plane. Equation (2) now may be solved to give the

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