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On the Evolution of Nearly Circular Vortex Patches

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Abstract. Recently, the classical problem of the evolution of patches of constant vorticity was reformulated as an evolution equation for the boundary of the patch. We study this equation in the neighborhood of the circular vortex patch and introduce a hierarchy of area-preserving nonlinear approximate equations. The first of these equations is shown to have a rich rigid structure: it possesses an exhaustive increasing sequence of linear invariant manifolds of arbitrarily large finite dimensions. On each of these manifolds the equation can be written as an explicit finite system of ordinary differential equations. Solutions of these ODEs, starting from arbitrarily small neighborhoods of the circular vortex patch, are shown to blow up.

Introduction

Recently, Zabusky et al. [9] and Majda [5] reformulated the classical hydrodynamical problem of the evolution of patches of constant vorticity as an evolution equation for the boundary of the patch. In [5] Majda conjectured that singularities will develop spontaneously in some solutions of the vortex patch equation (VPE). On the other hand, the circular, uniformly rotating vortex patch is a solution of the VPE whose linear marginal stability [4] and even mildly nonlinear stability [7] are well known. Of course, these known stability results are not strong enough to preclude instability and subsequent blow up of quantities such as the length or curvature of vortex patches whose initial shape is nearly circular. In this paper we study the evolution of nearly circular vortex patches. We devise a hierarchy of nonlinear area-preserving approximate equations. These equations intertwine with the equations obtained by retaining finitely many terms in the Taylor expansion near the circle of the nonlinearity. The first of the area-preserving approximate equations, a quadratic nonlocal correction of the linearization about the circular vortex patch, has some interest of its own. For every $n \geqq 0$ the linear manifold of trigonometric polynomials of degree at most n is invariant for this equation. This means that no cascade of "energy" from large scales to small scales occurs: high frequency modes are not excited during the evolution. We obtain blow up results for small data (i.e., nearly circular initial data) for this first approximate equation.