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Symmetric Harmonic Maps Between Spheres

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Abstract. Necessary and sufficient conditions for the existence of symmetric harmonic maps between spheres are established.

1. Introduction

In [S], Smith studied the problem of existence of harmonic maps between Euclidean spheres. The harmonic maps he constructed are of very special type. Starting from two homogeneous harmonic polynomial maps $f: S^p \to S^m$ of homogeneity k and $g: S^q \to S^n$ of homogeneity l, Smith sought for harmonic maps from S^{p+q+1} into S^{m+n+1} which are of the following form:

$$u(x,y) = \left(\sin a(t) f\left(\frac{x}{|x|}\right), \cos a(t) g\left(\frac{y}{|y|}\right)\right), \tag{1.1}$$

where $x \in R^{p+1}$, $y \in R^{q+1}$, with $|x|^2 + |y|^2 = 1$, $t = \log(|x|/|y|)$, and $a(t)$ is a real function with range in $[0, \frac{\pi}{2}]$. It is proved that if $a(t)$ satisfies the following equation:

$$\ddot{a} + \frac{(p-1)e^{-t} - (q-1)e^t}{e^t + e^{-t}}\dot{a} + \frac{\lambda_2 e^t - \lambda_1 e^{-t}}{e^t + e^{-t}} \sin a \cos a = 0,$$

together with the conditions

$$0 \leqq a(t) \leqq \tfrac{\pi}{2}, \qquad \forall t \in R,$$

and

$$\lim_{t \to -\infty} a(t) = 0, \qquad \lim_{t \to \infty} a(t) = \tfrac{\pi}{2},$$

then the map defined in (1.1) is actually an analytic harmonic map. Here $\lambda_1 = k(k+p-1)$ and $\lambda_2 = l(l+q-1)$. It is obvious that such a map is homotopic to the join $f * g$ of f and g [which is defined by (1.1) with $\sin a(t) = |x|$ and $\cos a(t) = |y|$].

Smith proved that if $(p-1)^2 < 4\lambda_1$ and $(q-1)^2 < 4\lambda_2$ or $p = q$ and $\lambda_1 = \lambda_2$, then there exists a harmonic map homotopic to $f * g$. Quite recently, Ratto [R] showed that the same conclusion holds provided $\lambda_1 = p \leqq 5$. In this paper, we completely solve the problem of existence of harmonic maps of Smith's type. Our main result is the following: