Commun. Math. Phys. 117, 37–47 (1988)

Communications in Mathematical Physics

Canonical Partition Functions of Hamiltonian Systems and the Stationary Phase Formula

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Abstract. We show that there is a Symplectic Action of a Torus associated to Harmonic flows on the Cotangent Bundle of a semi-simple Lie Algebra. This allows to obtain a completely classical proof of the Gallavotti–Marchioro Formula by the method of the Stationary Phase.

We compute the Canonical Partition Function,

$$Z(\beta)=\int_{V^{2m}} \exp(-\beta H)\Omega \qquad (\beta>0)$$

for a class of integrable Hamiltonian systems. H is the Hamiltonian function, $\Omega=\omega^m/m!$ is the volume form associated to a symplectic form ω of a symplectic manifold V^{2m}. We compute this integral by a singularity analysis and by an adaptation of the Duistermaat–Heckman theorem [D–H] on the exactness of the stationary phase formula. The proof that Berline–Vergne [B–V] gave of this theorem can be adapted to our situation where V^{2m} is not compact and where the parameter β is positive.

We obtain, for instance, the expression of the Canonical Partition Function for integrable systems which are reductions of the Harmonic flow on the cotangent bundle of a semi-simple Lie algebra. To this purpose, we need to check that these systems are associated to an Hamiltonian action of the torus. We show that the Hamiltonian flows of the eigenvalues of the Lax matrix are all periodic.

Our result extends a formula which was precedingly obtained by Gallavotti and Marchioro [G–M] for the Calogero–Moser system with an external quadratic potential which corresponds to the root system of type A_{m-1}. Gallavotti and Marchioro got their result by considering the quantum system and then by taking a limit ($\hbar \to 0$).

On the contrary, our proof is purely classical; it illustrates how the Duistermaat–Heckman formula can be used also to compute thermodynamical sums.

1 Adaptation of the Stationary Phase Method

We consider the proof of [B. V.] of the theorem of the stationary phase of Duistermaat and Heckman.