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A New Characterization of Half-Flat Solutions to Einstein's Equation

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Abstract. A $3+1$ formulation of complex Einstein's equation is first obtained on a real 4-manifold M, topologically $\Sigma \times R$, where Σ is an arbitrary 3-manifold. The resulting constraint and evolution equations are then simplified by using variables that capture the (anti-) self dual part of the 4-dimensional Weyl curvature. As a result, to obtain a vacuum self-dual solution, one has just to solve one constraint and one "evolution" equation on a field of triads on Σ:

$$\operatorname{Div} V_{\mathbf{i}}^{a}=0 \quad \text{and} \quad \dot{V}_{\mathbf{i}}^{a}=\varepsilon_{\mathbf{ijk}}[V_{\mathbf{j}}, V_{\mathbf{k}}]^{a}, \quad \text{with} \quad \mathbf{i}\equiv 1,2,3\,,$$

where Div denotes divergence with respect to a fixed, non-dynamical volume element. If the triad is real, the resulting self-dual metric is real and positive definite. This characterization of self-dual solutions in terms of triads appears to be particularly well suited for analysing the issues of exact integrability of the (anti-) self-dual Einstein system. Finally, although the use of a $3+1$ decomposition seems artificial from a strict mathematical viewpoint, as David C. Robinson has recently shown, the resulting triad description is closely related to the hyperkähler geometry that (anti-) self-dual vacuum solutions naturally admit.

I. Introduction

Over the past decade, considerable work has been done on half-flat[1] solutions to Einstein equations both in the Euclidean and the complex regimes. It turns out that the half-flatness requirement simplifies Einstein's equation significantly, whence it is possible to obtain several interesting results. The most powerful ones are the following. First, using three different approaches, Newman, Penrose, and Pleban-

[1] A 4-metric g_{ab} will be said to be *half-flat* if its Riemann tensor is proportional to its dual. Note that, due to the Bianchi identity ${}^4R_{[abc]d}=0$, a half-flat metric is necessarily Ricci flat. Because the square of the duality operator equals $+1$ if g_{ab} is Euclidean, and -1 if it is Lorentzian, a real metric can be half-flat only if it has Euclidean signature. In this case, we shall say that the metric is *self dual* if its Riemann tensor equals its dual and *anti-self dual* if it equals minus its dual