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# The Riemannian Geometry of the Yang-Mills Moduli Space

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**Abstract.** The moduli space $\mathscr{M}$ of self-dual connections over a Riemannian 4-manifold has a natural Riemannian metric, inherited from the $L^2$ metric on the space of connections. We give a formula for the curvature of this metric in terms of the relevant Green operators. We then examine in great detail the moduli space $\mathscr{M}_1$ of $k=1$ instantons on the 4-sphere, and obtain an explicit formula for the metric in this case. In particular, we prove that $\mathscr{M}_1$ is rotationally symmetric and has "finite geometry:" it is an incomplete 5-manifold with finite diameter and finite volume.

## Introduction

The moduli spaces of self-dual connections on vector bundles over a Riemannian 4-manifold have been studied from two different viewpoints. Mathematicians have sought to understand the topology of these moduli spaces. Most notable here is the work of S. Donaldson showing that even a rudimentary knowledge of this topology can lead to important results about smooth 4-manifolds. Physicists, on the other hand, study these spaces because the semiclassical – or "instanton" – approximation to the Green functions of (Euclidean) quantum Yang-Mills theory is expressed in terms of integrals over the moduli spaces. The evaluation of such integrals requires a detailed description of the *metric* and the *volume form* of the moduli spaces. In this paper we investigate moduli spaces with the goal of describing them as concrete Riemannian manifolds.

The relevant Riemannian metric on the moduli space $\mathscr{M}$ is the "$L^2$ metric", defined as follows. First, the space of connections on a principal bundle $P$ is an affine space $\mathscr{A}$ whose tangent space is the space of 1-forms with values in an associated vector bundle $\mathrm{Ad}P$. The $L^2$ inner product of such forms defines a Riemannian metric on $\mathscr{A}$. This metric is invariant under the action of the gauge group $\mathscr{G}$, and splits the tangent bundle $T\mathscr{A}$ into $\mathscr{G}$-invariant "vertical" and

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