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Parallel Transport in the Determinant Line Bundle: The Zero Index Case

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Abstract. For a product family of invertible Weyl operators on a compact manifold X, we express parallel transport in the determinant line bundle in terms of the spectral asymmetry of a Dirac operator on $\mathbf{R} \times X$.

0. Introduction

Let X be a compact spin manifold of even dimension with spin bundle $S = S_+ \oplus S_- \to X$, and let $E \to X$ be the hermitian vector bundle over X. Let $\bar{S}$ and $\bar{E}$ be the pullbacks of S and E to $\mathbf{R} \times X$ with the induced inner products, and let $\bar{\nabla}^E$ be a connection on $\bar{E}$. Thus $\bar{\nabla}^E = d_{\mathbf{R}} + \theta + \nabla^E_{(\cdot)}$, where $\theta \in \Omega^1(\mathbf{R}) \otimes C^\infty(X, \operatorname{End} E)$ and for each $y \in \mathbf{R}$, ∇^E_y is a connection of $E \to X$. Let ∂_y, $y \in \mathbf{R}$, be the Weyl operators ∂_y: $L^2(X, S_+ \otimes E) \mapsto L^2(X, S_- \otimes E)$ coupled to the connection ∇^E_y and the (y-independent) metric on X, and let $\nabla \partial = d_{\mathbf{R}} \partial + [\theta, \partial]$. Let H be the formally self adjoint Dirac operator on $L^2(\mathbf{R} \times X, \bar{S} \otimes \bar{E})$ coupled to $\bar{\nabla}^E$ and the product metric on $\mathbf{R} \times X$. Thus

$$H = \begin{pmatrix} i\left(\dfrac{\partial}{\partial y} + \theta\left(\dfrac{\partial}{\partial y}\right)\right) & \partial_y^\dagger \\ \partial_y & -i\left(\dfrac{\partial}{\partial y} + \theta\left(\dfrac{\partial}{\partial y}\right)\right) \end{pmatrix}.$$

Assume that for all $y \in \mathbf{R}$, ∂_y is invertible (so that $\operatorname{ind} \partial_y = 0$), and that for $|y|$ large, $\theta = 0$ and $d\nabla^E/dy = 0$. The main result of this paper is the formula

$$\exp \int_R \operatorname{Tr} \partial^{-1} \nabla \partial = \left(\frac{\det \partial_\infty^\dagger \partial_\infty}{\det \partial_{-\infty}^\dagger \partial_{-\infty}} \right)^{1/2} \exp \pi i (\eta(H) + \dim \operatorname{Ker} H). \tag{0.1}$$

Here $\det \partial_y^\dagger \partial_y$ is the determinant of $\partial_y^\dagger \partial_y$, given formally as the product of the

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