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Non-Representability of Cohomology Classes by Bi-Invariant Forms (Gauge and Kac–Moody Groups)

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Abstract. We give a necessary topological condition on a cohomology class of any Lie group $\mathscr{G}$, modelled on a Fréchet space, to be representable by a bi-invariant form on $\mathscr{G}$. As a corollary, we show that if $\Pi_{2d}(\mathscr{G})\bigotimes_{\mathbb{Z}}\mathbb{R} \neq 0$ for some $d > 0$, then there exists a cohomology class in $H^{2d}(\mathscr{G}, \mathbb{R})$ which cannot be represented by any bi-invariant form. In particular, we conclude that there are 'many' cohomology generators, in general, in the case of gauge groups and also Kac–Moody groups which cannot be represented by bi-invariant forms, although, very often, they are representable by left invariant forms.

Introduction

Using a mixture of (very simple) topological and geometrical arguments, we show that certain cohomology classes of infinite-dimensional Lie groups (modelled on Fréchet spaces) cannot be represented by bi-invariant forms.

Our main (and the only) theorem gives a necessary topological condition on a cohomology class, of a fairly arbitrary infinite dimensional group $\mathscr{G}$, to be representable by bi-invariant forms. An interesting corollary of the theorem is that if $x \in H^{2d}(\mathscr{G}, \mathbb{R})$, with $d > 0$ and x is not decomposable (i.e. $x \notin H^+(\mathscr{G}, \mathbb{R}) \cdot H^+(\mathscr{G}, \mathbb{R})$) (such a x always exists if $\Pi_{2d}(\mathscr{G})\bigotimes_{\mathbb{Z}}\mathbb{R} \neq 0$) then x cannot be represented by bi-invariant forms.

We apply this corollary to the particular (and important) examples of based loop groups, gauge groups and Kac–Moody groups to conclude that these groups, often, have many cohomology generators which cannot be represented by bi-invariant forms, although, in many cases, they can be represented by left invariant forms.

1. Definition. Let M be a smooth ($= C^\infty$) Fréchet manifold [M]. By $\Delta_\infty(M)$, we mean the *smooth singular chain-complex*/$\mathbb{Z}$ of M. More explicitly; by a smooth singular n-simplex in M, we mean a continuous map $s: \Delta^n = \Big\{(t_1, \ldots, t_n) \in \mathbb{R}^n: t_i \geqq 0$ and