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Classical and Quantum Mechanical Systems of Toda-Lattice Type

III. Joint Eigenfunctions of the Quantized Systems

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Abstract. In a previous paper it was shown that certain Schrödinger operators $H=\Delta-V$ on $\mathbb{R}^\ell$ such as the Hamiltonians for the quantized one-dimensional lattice systems of Toda type (either non-periodic or periodic) are part of a family of mutually commuting differential operators $H=L_1, \ldots, L_\ell$ on $\mathbb{R}^\ell$. The potential V in these cases is associated with a finite root system of rank ℓ, and the top-order symbols of the operators L_i are a set of functionally independent polynomials that generate the polynomial invariants for the Weyl group W of the root system. In this paper it is proved that the spaces of joint eigenfunctions for the family of operators L_i have dimension $|W|$. In the case of the periodic Toda lattices it is shown that the Hamiltonian has only bound states. An integrable holomorphic connection with periodic coefficients is constructed on a trivial $|W|$-dimensional vector bundle over $\mathbb{C}^\ell$, and it is shown that the joint eigenfunctions correspond exactly to the covariant constant sections of this bundle. Hence the eigenfunctions can be calculated (in principle) by integrating a system of ordinary differential equations. These eigenfunctions are holomorphic functions on $\mathbb{C}^\ell$, and are multivariable generalizations of the classical Whittaker functions and Mathieu functions. A generalization of Hill's determinant method is used to analyze the monodromy of the connection.

Contents

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