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Note on Loss of Regularity for Solutions of the $3-D$ Incompressible Euler and Related Equations*

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Abstract. One of the central problems in the mathematical theory of turbulence is that of breakdown of smooth (indefinitely differentiable) solutions to the equations of motion. In 1934 J. Leray advanced the idea that turbulence may be related to the spontaneous appearance of singularities in solutions of the $3-D$ incompressible Navier-Stokes equations. The problem is still open. We show in this report that breakdown of smooth solutions to the $3-D$ incompressible slightly viscous (i.e. corresponding to high Reynolds numbers, or "highly turbulent") Navier-Stokes equations cannot occur without breakdown in the corresponding solution of the incompressible Euler (ideal fluid) equation. We prove then that solutions of distorted Euler equations, which are equations closely related to the Euler equations for short term intervals, do breakdown.

Introduction

The purpose of this paper is twofold: first to discuss the relationship between the breakdown of smooth solutions to incompressible three-dimensional Euler and Navier-Stokes equations; and secondly to present blow-up results for distorted Euler equations.

Both the Navier-Stokes equations and the Euler equations possess local (in time) smooth solutions. Moreover, as the viscosity vanishes the solutions to the Navier-Stokes equations converge uniformly on a short time interval to the solution of the Euler equation [5, 7]. Adapting the method of Kato [5] and using a very simple ODE lemma, we prove in Sect. 1 that as long as the solution to the Euler equation is smooth the solutions to slightly viscous Navier-Stokes equations with the same initial data are smooth.

Sections 2 and 3 are devoted to blow-up results for distorted Euler equations. Differentiating the Euler equations one obtains a quadratic equation for the

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