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The Spectrum of a Schrödinger Operator in $L_p(\mathbb{R}^\nu)$ is p-Independent

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Abstract. Let $H_p = -\frac{1}{2}\Delta + V$ denote a Schrödinger operator, acting in $L_p(\mathbb{R}^\nu)$, $1 \leqq p \leqq \infty$. We show that $\sigma(H_p) = \sigma(H_2)$ for all $p \in [1, \infty]$, for rather general potentials V.

Introduction. In [12, 13], B. Simon conjectured that $\sigma(H_p)$ is p-independent, where $H_p = -\frac{1}{2}\Delta + V$ is a general Schrödinger operator in $L_p(\mathbb{R}^\nu)$. Partial results on this problem are contained in Simon [12], Sigal [10], Hempel, Voigt [5].

In the notations of Sect. 1, our main result reads as follows.

Theorem. *Let* $V = V_+ - V_-$, $V_\pm \geqq 0$, *where* V_+ *is admissible, and* $V_- \in \hat{K}_\nu$ *with* $c_\nu(V_-) < 1$. *Then* $\sigma(H_p) = \sigma(H_2)$ *for* $1 \leqq p \leqq \infty$.

In addition, if λ *is an isolated eigenvalue of finite algebraic multiplicity* k *of* H_p, *for some* $p \in [1, \infty]$, *then the same is true for all* $p \in [1, \infty]$.

The proof of this result is contained in Propositions 2.1, 3.1, and 2.2.

In Sect. 2 we prove the inclusion $\sigma(H_2) \subset \sigma(H_p)$, following ideas of Simon and Davies.

In Sect. 3 we show that the integral kernel of $(H_2 - z)^{-n}$, for $n \in \mathbb{N}$, $n > \nu/2$, defines an analytic $\mathscr{B}(L_p(\mathbb{R}^\nu))$-valued function on $\rho(H_2)$, which coincides with $(H_p - z)^{-n}$ for z real and sufficiently negative. This implies $\sigma(H_p) \subset \sigma(H_2)$, by unique continuation.

A different situation, where an integral kernel determines operators with *p-dependent* spectrum, can be found in Jörgens [6; IV, Aufg. 12.11 (b)]; note that the kernel in Jörgens' example is the resolvent kernel of the differential operator

$$-\frac{d}{dx}x^2\frac{d}{dx} \quad \text{on} \quad (0, \infty), \quad \text{at } z = -2.$$

1. Schrödinger Operators in $L_p(\mathbb{R}^\nu)$

First we recall briefly several facts concerning the semigroup associated with the heat equation. For brevity, we shall write L_p instead of $L_p(\mathbb{R}^\nu)$, in the sequel