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The Chern Classes of Sobolev Connections

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Abstract. Assume F is the curvature (field) of a connection (potential) on $\mathbf{R}^4$ with finite L^2 norm $\left(\int_{\mathbf{R}^4} |F|^2 dx < \infty\right)$. We show the chern number $c_2 = 1/8\pi^2 \int_{\mathbf{R}^4} F \wedge F$ (topological quantum number) is an integer. This generalizes previous results which showed that the integrality holds for F satisfying the Yang–Mills equations. We actually prove the natural general result in all even dimensions larger than 2.

0. Introduction

All solutions of the Yang–Mills equations on $\mathbf{R}^4$ with finite action actually arise from connections defined on $\mathbf{R}^4 \cup (\infty) = S^4$ [1,2]. This implies that the chern numbers of these connections are the chern numbers of a bundle over S^4, and hence are integers. It seems to be a question of general interest whether this result holds for arbitrary connections on $\mathbf{R}^4$ with finite energy [3]. Schlafley showed this is indeed true if the curvature or field $|F|$ has growth at most $(r^2 \log r)^{-1}$ [4]. We prove that finite energy $\int_{\mathbf{R}^4} |F|^2 dx$ is sufficient. We prove general n-dimensional results. We assume throughout the paper that G is a compact Lie group with bi-invariant metric and $\mathfrak{g}$ is the Lie algebra for G.

Theorem. *Let $A_j \in L^{n/2}_{1,\mathrm{loc}}(\mathbf{R}^n, \mathfrak{g})$, $j = 1, 2, \ldots, n > 2$ and let $F = F_A = dA + A \wedge A$ be the curvature of the connection $d + A$. If n is even, $n \neq 2$, and $\int_{\mathbf{R}^n} |F|^{n/2} dx < \infty$, then the chern number arising from a representation $\rho: G \to \mathrm{SU}(N)$ is integral.*

The proof is somewhat lengthy, and could be shortened considerably for the case A_j smooth. However, it seemed worthwhile to treat the most general case, $A_j \in L^{n/2}_{1,\mathrm{loc}}$, for the purpose of completeness. The various technical theorems we use to handle non-smooth A_j have interesting features and possible applications elsewhere. The main idea of the proof is to choose a good gauge near (∞). This relies on an earlier theorem on the existence of good (Coulomb) gauges [5]. The idea for the proof arose from conversations with L. M. Sibner about the removable singularities theorem in dimension 3 [6].