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Scaling of Mandelbrot Sets Generated by Critical Point Preperiodicity

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Abstract. Let $z \to f_\mu(z)$ be a complex holomorphic function depending holomorphically on the complex parameter μ. If, for $\mu = 0$, a critical point of f_0 falls after a finite number of steps onto an unstable fixed point of f_0, then, in the parameter space, near 0, an infinity of more and more accurate copies of the Mandelbrot set appears. We compute their scaling properties.

In several numerical experiments, the dynamics of rational maps has been studied as the coefficients of the map vary. It has been observed [M, DH, S, CGS, C] that remarkably precise copies of the standard Mandelbrot set $\mathscr{M}$ appear in the corresponding plots. In their recent work [DH], Douady and Hubbard explain this phenomenon by the local occurrence of "polynomial-like maps of degree 2." In this paper, we elaborate on this explanation by showing that the mechanism of "quadratification by large order iteration" [G, EEW] produces infinite sequences of copies of $\mathscr{M}$, obeying simple scaling properties.

We describe now the setting of a pre-periodic critical point, but immediately restrict our discussion to the case of period one. We consider a complex holomorphic function $(z, \mu) \to f_\mu(z)$ over $H \times D_M \subset \mathbb{C}^2$, where H is a domain in $\mathbb{C}$ and $D_M = \{\zeta \in \mathbb{C} \mid |\zeta| < M\}$, $M > 0$. We denote by f_μ the function $z \to f_\mu(z)$, and $D^{\alpha\beta} f_\mu = \partial_z^\alpha \partial_\mu^\beta f_\mu$ [1], $f'_\mu = \partial_z f_\mu$, etc. We make the following assumptions:

A1: f_0 has a non-degenerate critical point c.

A2: f_0 has an unstable fixed point u, $f'_0(u) = \gamma$, $|\gamma| > 1$. (The restriction of f_0 to a sufficiently small neighbourhood of u has a unique inverse which we denote f_0^{-1}.)

A3: (critical point preperiodicity) $f_0^Q(c) = u$,

$$(f_0^Q)''(c) \neq 0, \quad \text{for some} \quad Q \geqq 1 .$$

A4: There is a sequence $\{x_n\}$ of points in H accumulating at u such that

$$f_0(x_{n+1}) = x_n, f'_0(x_n) \neq 0, \quad \text{for} \quad n = 1, 2, \ldots,$$

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1 By abuse of notation, we write $D^{\alpha\beta} f_0$ instead of $D^{\alpha\beta} f_\mu|_{\mu=0}$