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Convergence of the Quantum Boltzmann Map

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Abstract. We consider a non-linear map on the space of density matrices, which we call the Boltzmann map τ. It is the composition of a doubly stochastic map T on the space of n-body states, and the conditional expectation onto the one-body space. When T is ergodic, then the iterates of τ take any initial state to the uniform distribution. If the energy levels are equally spaced, and T conserves energy and is ergodic on each energy shell, then iterates of τ take any initial state of finite energy to a canonical distribution.

1. Introduction

(1.1) This paper is the quantum version of [1]. Let $\mathscr{H}$ be a Hilbert space with $\dim\mathscr{H}=N\leqq\infty$. A (normal) state ϱ is then a positive operator with unit trace. We denote the set of trace-class operators by $\mathscr{B}(\mathscr{H})_1$ and the normal[1] states by $\sigma(\mathscr{H})$. A stochastic map is a linear map T from $\mathscr{B}(\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ mapping $\sigma(\mathscr{H})$ to itself and preserving the trace: $\mathrm{Tr}(T\varrho)=\mathrm{Tr}\,\varrho$, $\varrho\in\mathscr{B}(\mathscr{H})_1$. A doubly stochastic map is a stochastic map T such that $T1_N=1_N$, where 1_N is the identity on $\mathscr{H}$ [4].

A unitary or anti-unitary conjugation $\varrho\mapsto T\varrho=U\varrho U^{-1}$ is doubly stochastic, as is any convex combination of such maps.

(1.2) Let $\mathscr{K}$ be a Hilbert space, the one-particle space, and

(1.3) let $\mathscr{H}=\mathscr{K}\otimes\ldots\otimes\mathscr{K}$ (n factors) be the n-particle space.

We shall be interested in a doubly stochastic map $T:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{H})$ that preserves the symmetry under permutations of the factors $\mathscr{K}$. To such a T we define the corresponding *Boltzmann map* τ to be the composition of maps:

(1.4) $$\varrho\mapsto\varrho\otimes\ldots\otimes\varrho\mapsto T(\varrho\otimes\ldots\otimes\varrho)\mapsto\mathrm{Tr}_{2\ldots n}T(\varrho\otimes\ldots\otimes\varrho)=\tau(\varrho).$$

Here, $\mathrm{Tr}_{2\ldots n}$ means the trace over the second, third, ..., n^{th} factors $\mathscr{K}$. Obviously, (1.4) defines a non-linear map $\tau:\sigma(\mathscr{K})\to\sigma(\mathscr{K})$.

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1 Normal in the sense [2] of linear functionals on the W^*-algebra $\mathscr{B}(\mathscr{H})$, not in the sense of [3]