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An Invariant Measure for the Equation $u_{tt} - u_{xx} + u^3 = 0$

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Abstract. Numerical studies of the initial boundary-value problem of the semilinear wave equation $u_{tt} - u_{xx} + u^3 = 0$ subject to periodic boundary conditions $u(t, 0) = u(t, 2\pi)$, $u_t(t, 0) = u_t(t, 2\pi)$ and initial conditions $u(0, x) = u_0(x)$, $u_t(0, x) = v_0(x)$, where $u_0(x)$ and $v_0(x)$ satisfy the same periodic conditions, suggest that solutions ultimately return to a neighborhood of the initial state $u_0(x)$, $v_0(x)$ after undergoing a possibly chaotic evolution. In this paper an appropriate abstract space is considered. In this space a finite measure is constructed. This measure is invariant under the flow generated by the Hamiltonian system which corresponds to the original equation. This enables one to verify the above "returning" property.

0. Introduction

During the Sixth I. G. Petrovskii memorial meeting of the Moscow Mathematical Society in January 1983 Professor V. E. Zakharov proposed the following problem. Numerical experiments demonstrated that the equation

$$u_{tt} - u_{xx} + u^3 = 0 \tag{0.1}$$

with periodic boundary conditions $u(t, 0) = u(t, 2\pi)$, $u_t(t, 0) = u_t(t, 2\pi)$ possesses the "returning" property, i.e. solutions appear to be very close to the initial state $u(0, x) = u_0(x)$, $u_t(0, x) = v_0(x)$, where the initial functions satisfy the above boundary conditions, after some time of rather chaotic evolution. The problem is to explain this phenomenon. According to the classical Poincaré theorem every flow which preserves a finite measure has the returning property modulo a set of measure zero. The aim of this paper is to build such a measure for the flow

$$\Phi(t)(u_0(x), v_0(x)) = (u(t, x), v(t, x)),$$

where $u(t, x)$ is the solution of (0.1), $v(t, x) = u_t(t, x)$, where the solution u satisfies the initial data $u(0, x) = u_0(x)$, $u_t(0, x) = v_0(x)$. The Eq. (0.1) can be rewritten as a Hamiltonian system

$$\left.\begin{aligned} u_t &= \delta H/\delta v \\ v_t &= -\delta H/\delta u \end{aligned}\right\} \tag{0.2}$$