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The Surfboard Schrödinger Equations

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Abstract. We study the large time behavior of solutions of time dependent Schrödinger equations $i\partial u/\partial t = -(\frac{1}{2})\Delta u + t^\alpha V(x/t)u$ with bounded potential $V(x)$. We show that (1) if $\alpha > -1$, all solutions are asymptotically free as $t \to \infty$, (2) if $\alpha \leqq -1$ a solution becomes asymptotically free if and only if it has the momentum support outside of supp V for large time, (3) if $-1 \leqq \alpha < 0$ all solutions are still asymptotically "modified free" as $t \to \infty$ and that (4) if $0 \leqq \alpha < 2$, for each local minimum x_0 of $V(x)$, there exist solutions which are asymptotically Gaussians centered at $x = tx_0$ and spreading slowly as $t \to \infty$.

1. Introduction

Several years ago Kuroda and Morita [6] proposed the study of the Schrödinger equations of the form

$$i\partial u/\partial t = -(\tfrac{1}{2})\Delta u + t^\alpha V(x/t)u, \quad t \geqq 1, \quad x \in \mathbb{R}^n, \tag{1.1}$$

in conjunction with their study of the equations

$$i\partial u/\partial t = -(\tfrac{1}{2})\Delta u + t^\alpha v(x/t^\beta)u, \quad t \geqq 1, \quad x \in \mathbb{R}^n, \tag{1.2}$$

with $\beta \neq 1$. Equation (1.1) is considered in the Hilbert space $L^2(\mathbb{R}^n) = \mathscr{H}$ of square integrable functions and was named the surfboard Schrödinger equation because of its obvious pictorial analogy with the motion of a surfboard: the potential spreads at the same rate as a free wave packet. In this paper we shall study the asymptotic behavior of the solution of (1.1) with $\alpha < 2$, and show the following results. We write the propagator for Eq. (1.1) as $U(t,s)$ and $H_0 = -(\frac{1}{2})\Delta$.

(I) If $\alpha < -1$, then for every $u \in \mathscr{H}$, the strong limit

$$\lim_{t\to\infty} U(1,t)\exp(-i(t-1)H_0)u = W_+ u \tag{1.3}$$

exists and the wave operator W_+ is unitary.

(II) If $\alpha \geqq -1$, the limit (1.3) exists if and only if $\mathfrak{F}u \in L^2((\text{supp } V)^c)$, the elements in $\mathscr{H}$ whose essential supports are in the complement of the support of V.

(III) If $0 > \alpha \geqq -1$, the modified wave operator still exists and is unitary.