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Minimum Action Solutions of Some Vector Field Equations

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Abstract. The system of equations studied in this paper is $-\Delta u_i = g^i(u)$ on $\mathbb{R}^d$, $d \geqq 2$, with $u: \mathbb{R}^d \to \mathbb{R}^n$ and $g^i(u) = \partial G/\partial u_i$. Associated with this system is the action, $S(u) = \int \{\frac{1}{2}|\nabla u|^2 - G(u)\}$. Under appropriate conditions on G (which differ for $d=2$ and $d \geqq 3$) it is proved that the system has a solution, $u \not\equiv 0$, of finite action and that this solution also minimizes the action within the class $\{v$ is a solution, v has finite action, $v \not\equiv 0\}$.

I. Introduction

The purpose of this paper is to demonstrate the existence of solutions to a class of systems of partial differential equations that arises in several branches of mathematical physics (e.g. calculating lifetimes of metastable states, estimates of large order behavior of perturbation theory, Ginzburg-Landau theory, density of states in disordered systems). The systems to be considered are of the form

$$-\Delta u_i(x) = g^i(u(x)), \qquad i = 1, \ldots, n. \tag{1.1}$$

Furthermore, it will be shown that among the nonzero solutions to (1.1) there is one that minimizes the action, $S(u)$, associated with (1.1).

The meaning of the quantities in (1.1) is the following: $u \equiv (u_1, \ldots, u_n) \in \mathbb{R}^n$ and each $u_i: \mathbb{R}^d \to \mathbb{R}$ with $d \geqq 2$. We require that $u_i(x) \to 0$ as $|x| \to \infty$ in a weak sense described below (namely $u \in \mathscr{C}$). (Note: In some applications it is required that $u(x) \to c =$ constant as $|x| \to \infty$ but, by redefining $u \to u - c$ and by redefining g^i, the problem can be reduced to the $u(x) \to 0$ case.) The n functions $g^i: \mathbb{R}^n \to \mathbb{R}$ are the gradients of some function $G \in C^1(\mathbb{R}^n \backslash \{0\})$, namely

$$\begin{aligned} g^i(u) &= \partial G(u)/\partial u_i, \qquad && u \neq 0, \\ g^i(u) &= 0, && u = 0, \end{aligned} \tag{1.2}$$

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