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Euler Equations on Finite Dimensional Lie Algebras Arising in Physical Problems

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Abstract. Real physical problems are presented in which Euler equations on Lie algebras of arbitrarily high finite dimension arise. A new integrable case of rotation of a magnetized rigid body in constant gravitational and magnetic fields is found. It generalizes the Kowalewski classical integrable case.

1. Physical Problems Related to Euler Equations on Finite-Dimensional Lie Algebras

The classical equations describing rotations of a free rigid body around its center of mass were derived by Euler in 1758 [1],

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega}, \tag{1.1}$$

where $\mathbf{M}$ and $\boldsymbol{\omega}$ are the angular momentum and angular velocity vectors; the equality relating their coordinates is $M_i = \sum_{k=1}^{3} I_{ik}\omega_k$, where I_{ik} are components of the inertia tensor of the rigid body. From the modern point of view, Eqs. (1.1) are given in the space conjugate to the Lie algebra SO(3); they are Hamiltonian equations on the orbits $M_1^2 + M_2^2 + M_3^2 = \text{const}$. Arnold [3] proposed the following extension of Euler equations (1.1) for arbitrary Lie algebras,

$$\dot{\mathbf{M}} = \mathrm{ad}^*_{a(\mathbf{M})}\mathbf{M}, \tag{1.2}$$

where the vector $\mathbf{M}$ belongs to a space L^* conjugate to the Lie algebra L, and $a(\mathbf{M})$ is a linear self-conjugate (with respect to the natural pairing) operator from L^* to L. Equations (1.2) are Hamiltonian equations on orbits $\mathcal{O}$ of the co-adjoint representation (in the space L^*) of the Lie group L, associated with the Lie algebra L. The Hamiltonian for Eq. (1.2), $H = \frac{1}{2}(a(\mathbf{M}), \mathbf{M})$, is necessarily a homogeneous second-order polynomial of components of the vector $\mathbf{M}$. Until recently, there was an opinion that Euler equations in finite-dimensional Lie algebras which can be