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Fixed Points of Feigenbaum's Type for the Equation $f^p(\lambda x) \equiv \lambda f(x)$

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Abstract. Existence and hyperbolicity of fixed points for the map $\mathcal{N}_p: f(x) \to \lambda^{-1} f^p(\lambda x)$, with f^p p-fold iteration and $\lambda = f^p(0)$ are given for p large. These fixed points come close to being quadratic functions, and our proof consists in controlling perturbation theory about quadratic functions.

1. Introduction

The main theme of this paper is another manifestation of the observation "highly iterated maps are quadratic functions," made by Jakobson [1], Milnor [2], Guckenheimer [3], and Benedicks and Carleson [4]. We shall elaborate on this idea and use it to give a simple proof of Feigenbaum universality for certain classes of functions.

We consider maps in a class of function $\mathfrak{D}_p$ which we shall describe now informally and in more detail below. We shall say that $f \in \mathfrak{D}_p$ if $f: [-1, 1] \to [-1, 1]$, $f \in \mathscr{C}^2$ (in fact, we shall work with analytic functions below), $f(0) = 1$, $f''(0) < 0$, and, most importantly, f permutes cyclically p disjoint intervals $J_0, J_1, \ldots, J_p$ with $0 \in J_0$. The intervals are supposed to be arranged as follows and

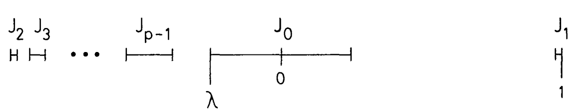

Fig. 1

the endpoints of J_0 are $f^p(0) < 0 < -f^p(0)$. Under these circumstances, setting $\lambda = f^p(0)$, one can show, see Collet, Eckmann, and Lanford [5], that $\mathcal{N}_p f(x) = \frac{1}{\lambda} f^p(\lambda x)$ is again a map of $[-1, 1]$ to itself and $(\mathcal{N}_p f)(0) = 1$. The contention is now: *If $f \in \mathfrak{D}_p$ is not too far from being a quadratic function, then the same is true for $\mathcal{N}_p f$.* Following Guckenheimer, we measure this deviation from being

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