Commun. Math. Phys. 93, 285–300 (1984)

Communications in Mathematical Physics

Characteristic Exponents for a Viscous Fluid Subjected to Time Dependent Forces

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Abstract. We consider a viscous incompressible fluid enclosed in a bounded region of $\mathbb{R}^2$ or $\mathbb{R}^3$, and subjected to time dependent forces. Using bound state estimates for the Schrödinger operator, we obtain rigorous bounds for the characteristic exponents, entropy (Kolmogorov-Sinai invariant), and Hausdorff dimension of attracting sets. Our methods are of potential use for more general time evolutions described by nonlinear partial differential equations.

1. Introduction

In an earlier paper (Ruelle [32]) some rigorous inequalities on the characteristic exponents for the Navier-Stokes time evolution have been obtained. These inequalities were based on estimates for the eigenvalues of Schrödinger operators, and have been subsequently improved by Lieb [18]. Using other methods, Constantin and Foias [4] have also investigated the characteristic exponents for Navier-Stokes (in the 2-dimensional case).

From the estimates on characteristic exponents one obtains rigorous bounds on the Kolmogorov-Sinai invariant (rate of creation of information) and the Hausdorff dimension of attracting sets. Such estimates were given in [32] and [4]. We indicate below the best bounds currently known, based on the results of Lieb [18] (see Eqs. (14)–(17)). This improves in particular the bounds on the Hausdorff dimension of attracting sets obtained by Constantin and Foias [4]. Explicit results are also obtained for a 2-dimensional convection problem.

One novelty of the present paper is to lift the requirement that the forces acting on the fluid be time independent.

2. Definition of the Characteristic Exponents

We consider a time evolution equation

$$\frac{d}{dt}x = F(x, t) \tag{1}$$