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Borel Summability of the Unequal Double Well

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Abstract. Unlike the $\varepsilon=0$ case, the perturbation series of the unequal double well $p^2+x^2+2gx^3+g^2(1+\varepsilon)x^4$ are Borel summable to the eigenvalues for any $\varepsilon>0$.

The best known example (see e.g. [13, Sect. XII.4]) of a non-Borel summable perturbation series is represented by the Rayleigh-Schrödinger perturbation expansion (hereafter RSPE) of the standard double well oscillator $H(g)=p^2+x^2+x^2(1+gx)^2$ in $L^2(\mathbb{R})$, $g\in\mathbb{R}$. This fact is of course due to the instability of the eigenvalues as $g\to 0$, i.e. to their asymptotic degeneracy as $g\to 0$. However there are examples, such as the Herbst and Simon [5] one, $K(g)=p^2+x^2(1+gx)^2-2gx-1$, in which there is stability but no Borel summability to the eigenvalues. Hence, also on account of recent investigations on Borel summability in four dimensional field theories [6, 7], it could be interesting to relate the lack of summability to some other more subtle physical mechanism of well defined meaning also in a more general context. To this end, T. Spencer has suggested considering the following "unequal" double well oscillator

$$H(g,\varepsilon)=p^2+x^2(1+gx)^2+\varepsilon g^2x^4\,, \tag{1}$$

which in the limit $g\to 0$ has an infinite action instanton for any $\varepsilon\geqq 0$. (A standard reference for the notion of instanton in problems of this type is Coleman [1]; additional discussion can be found in [2, 11].) This model could in addition have some interest in itself: as a matter of fact, in some sense it represents the slightest modification of the non-summable example, and it is natural to ask to what extent the non-summability as "accidental," i.e. how sensitive is its dependence on the choice of the parameters in $H(g)$? Furthermore it can be easily proved through the Hunziker-Vock technique [8] that any eigenvalue E of $H(0,\varepsilon)\equiv H(0)=p^2+x^2$ is stable for $g\in\mathbb{R}$ small as an eigenvalue of $H(g,\varepsilon)$, $\varepsilon>0$, because the second minimum of $V(g,\varepsilon)\equiv x^2(1+gx)^2+\varepsilon g^2x^4$ tends to $+\infty$ as $g\to 0$, $\varepsilon>0$.