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Stability in Yang-Mills Theories

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Abstract. At a solution of the Yang-Mills equations on S^4, or the Yang-Mills-Higgs equation on $\mathbb{R}^3$, the hessian of the action functional defines a natural second order, elliptic operator. The number of negative eigenvalues of this operator is bounded below by a multiple of the relevant topological charge. The proof of this assertion requires a relative index theorem for Dirac-type operators on $\mathbb{R}^n$, $n \geqq 3$.

I. Introduction

The Yang-Mills equations on S^4 and the static Yang-Mills equations on $\mathbb{R}^4$ (the Yang-Mills-Higgs equations on $\mathbb{R}^3$) are both variational equations of functionals on topologically interesting spaces. A solution to the equations is a critical point of the functional. At a critical point, the differential of the functional is zero. The second variation of the functional is called the hessian, and it is a bilinear form on a suitably defined Hilbert space. The index of the critical point is defined to be the number of eigenvectors of the hessian which have negative eigenvalues. The purpose of this article is to prove that because of topological considerations, certain values of the index do not occur for the Yang-Mills functional on S^4 and the Yang-Mills-Higgs functional on $\mathbb{R}^3$.

Consider first the SU(2) Yang-Mills equations on S^4 (cf. [1, 2] for reviews). The function space, $\mathfrak{B}$, is the space of isomorphism classes of pairs (P, A) where $P \to S^4$ is a principal SU(2)-bundle and A is a smooth connection on P. With respect to the C^∞-topology, $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ is the disjoint union of spaces $\mathfrak{B}_n$ which are indexed by $n \in \mathbb{Z}$. The integer n is minus the second Chern class of $P \times_{\mathrm{SU}(2)} \mathbb{C}^2$. (It is the physicist's instanton number.) Bourguignon et al. [3] have shown that every local minimum of the Yang-Mills functional, $\mathfrak{YM}$, on $\mathfrak{B}_n$ is (anti) self-dual. Non-minimal critical points of $\mathfrak{YM}$ have yet to be discovered, but if one exists, the following result applies:

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