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Non-Periodic and not Everywhere Dense Billiard Trajectories in Convex Polygons and Polyhedrons

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Abstract. This paper proves the existence of non-periodic and not everywhere dense billiard trajectories in convex polygons and polyhedrons. For any $n \geqq 3$ there exists a corresponding convex n-agon (for $n=3$ this will be a right triangle with a small acute angle), while in three-dimensional space it will be a prism, the n-agon serving as the base.

The results are applied for investigating a mechanical system of two absolutely elastic balls on a segment, and also for proving the existence of an infinite number of periodic trajectories in the given polygons.

The exchange transformation of two intervals is used for proving the theorems. An arbitrary exchange transformation of any number of intervals can also be modeled by a billiard trajectory in some convex polygon with many sides.

1. Introduction. Formulation of Theorems

Billiards in a polygon Q on the Euclidean plane $\mathbb{R}^2$ are a dynamical system. This system is defined by uniform motion of a point (particle) inside Q with elastic reflections at the polygon's boundary ∂Q, such that the tangential component of the velocity remains constant and the normal component changes sign. We will assume that the magnitude of the particle's velocity equals one.

The motion described is not limited in time if the particle does not fall into a vertex of Q. Otherwise, its motion is determined only till it falls into a vertex; this is a special case and we will not consider it.

The phase space $\mathscr{M} = \mathscr{M}(Q)$ of this dynamical system is a subset of the direct product $Q \times S^1$ (S^1 being the circle of unit velocities), which is obtained by identifying pairs (q, v) and (q, v') for $q \in \partial Q$, $v, v' \in S^1$ and

$$v - v' = 2(n, v)n, \tag{$*$}$$

where $n = n(q)$ is the external unit vector normal to ∂Q at point q.

A billiard system in a polyhedron is defined in the same way.