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The Random-Walk Representation of Classical Spin Systems and Correlation Inequalities

II. The Skeleton Inequalities

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Abstract. We use the random-walk representation to prove the first few of a new family of correlation inequalities for ferromagnetic φ^4 lattice models. These inequalities state that the finite partial sums of the propagator-resummed perturbation expansion for the 4-point function form an alternating set of rigorous upper and lower bounds for the exact 4-point function. Generalizations to $2n$-point functions are also given. A simple construction of the continuum φ_d^4 quantum field theory ($d<4$), based on these inequalities, is described in a companion paper.

1. Introduction

This paper is a continuation of the work begun in preceding papers [1–3], where a random-walk expansion due originally to Symanzik [4, 5] (see also [6, 7]) is employed to derive a variety of correlation inequalities (among other results) for lattice models in classical statistical mechanics. The main result of [2] (see also [3] for a variant of the proof) is the new correlation inequality

$$0 \geqq u_4(x_1, x_2, x_3, x_4) \geqq - \sum_{z, z', z''} \langle \varphi_{x_1} \varphi_z \rangle \langle \varphi_{x_2} \varphi_z \rangle J_{zz'} J_{zz''} \langle \varphi_{z'} \varphi_{x_3} \rangle \langle \varphi_{z''} \varphi_{x_4} \rangle - \text{two permutations} - \mathscr{E}, \tag{1.1}$$

where $\mathscr{E}$ is an extra term which turns out to be irrelevant in applications. This inequality implies [2, 3, 8–10] the *triviality* (i.e. Gaussianness) of the continuum limit for φ^4 or Ising models in dimension $d>4$. (For the Ising model this result was first obtained by Aizenman [8, 9], who proved a correlation inequality similar to (1.1) by graphical methods. A version of Aizenman's inequality also applies to the φ^4 model.)

In this paper we restrict attention to φ^4 models, and derive new correlation inequalities which will be (among other things) important ingredients in the proof of the *nontriviality* (i.e. non-Gaussianness) of the continuum limit for weakly coupled

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