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Regularity and Decay of Lattice Green's Functions*

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Abstract. In the paper we study a class of lattice, covariant Laplace operators with external gauge fields. We prove that these operators are positive and that their Green's functions decay exponentially. They also have regularity properties similar to continuous space Green's functions. All the bounds are uniform in the lattice spacing.

1. Introduction. Formulations of Theorems

In this paper we prove some properties of Green's functions for difference Laplace operators. They imply all the properties of Green's functions (propagators) and Gaussian actions used in the papers [1–3], thereby completing the proof of ultraviolet stability of $(\text{Higgs})_{2,3}$ models. But the range of applicability of these properties is much wider and a need of them appears in many mathematical problems of statistical physics and quantum field theory. In fact, the mathematical estimates have an intrinsic interest of their own, so we present them in a self contained paper.

The properties of Green's functions we are interested in are regularity properties and exponential decay. The difference Laplace operators are lattice approximations to second order elliptic differential operators, so the regularity properties of lattice Green's functions are similar to the properties of continuous space Green's functions. For example, if G is a lattice Green's function, then $\|Gf\|_2$, $\|\partial_\mu Gf\|_2$, $\|\partial_\mu \partial_\nu Gf\|_2 \leqq c\|f\|_2$, where ∂_μ is a difference derivative and $\|\cdot\|_2$ denotes L^2-norm on the lattice. Similar estimates hold for other norms, and we are especially interested in Hölder norms. Exponential decay is interpreted in physics as the existence of an "effective mass". In mathematical terms this means that there is a strictly positive lower bound for the inverse Green's operators. The simplest example is the operator $-\Delta + m^2$. A Green's function $C_{m^2} = (-\Delta + m^2)^{-1}$ for this operator, on continuous space and on a lattice, satisfies a bound of the form $|C_{m^2}(x,y)| \leqq O(1)|x-y|^{-d+2} e^{-m|x-y|}$. We are interested in proving similar bounds, or bounds derived from these, for more general operators.

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