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The Rotation Number for Finite Difference Operators and its Properties

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Abstract. We discuss a rotation number $\alpha(\lambda)$ for second order finite difference operators. If $k(\lambda)$ denotes the integrated density of states, then $k(\lambda)=2\alpha(\lambda)$. For almost periodic operators, $k(\lambda)$ is proved to lie in the frequency-module whenever λ is outside the spectrum; this yields a labelling of the gaps of the spectrum.

Introduction

We study in this paper the Jacobi matrices, acting on $\ell^2(\mathbb{Z})$:

$$(Hu)(n) = -u(n+1) - u(n-1) + V(n)u(n), \tag{1}$$

and more general second order finite difference operators defined later below. These operators have been of interest for a long time in mathematics, and also in physics where they appear in the tight binding approximation of one-dimensional condensed matter systems. They can be viewed as a finite difference analogue of the Schrödinger equation. On the other hand they have recently received renewed interest both in mathematics and physics particularly in the cases where the diagonal elements $V(n)$ are realizations of a sequence of independent random variables or when they constitute an almost periodic sequence. For reviews describing motivations and results in these two situations we refer, respectively, to references [10] and [15].

In the study of the continuous Schrödinger equation, one of the basic tools is the rotation[1] number $\alpha(\lambda)$; it follows from Sturm Liouville theory that it is equal to half the integrated density of states $k(\lambda)$. Recently Johnson and Moser [9] found the following remarkable result: if the potential $V(\cdot)$ of the continuous Schrödinger equation is an almost periodic function, $k(\lambda)=2\alpha(\lambda)$ lies in the

1 Our definition may differ by a factor 2π from the one used by other authors; same remark for the frequency module