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Kotani Theory for One Dimensional Stochastic Jacobi Matrices*

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Abstract. We consider families of operators, H_ω, on ℓ_2 given by $(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n)$, where V_ω is a stationary bounded ergodic sequence. We prove analogs of Kotani's results, including that for a.e. $\omega, \sigma_{\mathrm{ac}}(H_\omega)$ is the essential closure of the set of E where $\gamma(E)$ the Lyaponov index, vanishes and the result that if V_ω is non-deterministic, then σ_{ac} is empty.

1. Introduction

In a beautiful paper, Kotani [10] has proved three remarkable theorems about one-dimensional stochastic Schrödinger operators, i.e. operators of the form $-d^2/dx^2 + V_\omega(x)$ on $L^2(-\infty,\infty)$, where V_ω is a stationary bounded ergodic process. It is not completely straightforward to extend his proofs to the case where $-d^2/dx^2$ is replaced by a finite difference operator, and that is our goal in this note.

Explicitly, let (Ω,μ) be a probability measure space, T a measure preserving invertible ergodic transformation, and f a bounded measurable real-valued function. We define $V_\omega(n) = f(T^n\omega)$. We let H_ω be the operator on $\ell^2(Z)$

$$(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n).$$

Integrals over ω will be denoted by $E(\cdot)$.

Given a subset, J, of Z, we let Σ_J be the sigma–algebra generated by $\{V_\omega(n)\}_{n\in J}$. We say that the process is *deterministic* if $\Sigma_{-\infty} \equiv \bigcap_{j=1}^{\infty} \Sigma_{(-\infty,-j)}$ is up to sets of measure zero, $\Sigma_{(-\infty,\infty)}$; equivalently if $V_\omega(n)$ is a.e., a measurable function of $\{V_\omega(n)\}_{n\leqq 0}$. Otherwise it is *non-deterministic*. Almost periodic sequences are deterministic. Independent, identically distributed random variables are non-deterministic.

The Lyaponov index $\gamma(E)$ is defined, for example, in [1, 4]. It can be characterized as follows: For each complex E, for a.e. ω, any solution of $H_\omega u = Eu$ (in sequence sense) has $\lim_{n\to\infty} \frac{1}{n} \ln [|u(n)|^2 + |u(n+1)|^2]^{1/2}$ exists and it is either γ or $-\gamma$. It is an

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