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On the Construction of Monopoles

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Abstract. We show that any self-dual SU(2) monopole may be constructed either by Ward's twistor method, or Nahm's use of the ADHM construction. The common factor in both approaches is an algebraic curve whose Jacobian is used to linearize the non-linear ordinary differential equations which arise in Nahm's method. We derive the non-singularity condition for the monopole in terms of this curve and apply the result to prove the regularity of axially symmetric solutions.

1. Introduction

We shall be concerned in this paper with constructing solutions to the Bogomolny equations $D\Phi = *F$. Here F is the curvature of an SU(2) connection on $\mathbb{R}^3$, Φ (the Higgs field) is a section of the adjoint bundle, and we are seeking solutions for which $\|\Phi\| = 1 - kr^{-1} + O(r^{-2})$ as $r \to \infty$. These are particular solutions to the static, finite energy Yang-Mills-Higgs equations and we shall often refer to them simply as "monopoles".

There exist already two different approaches to constructing monopoles. One is due to R. S. Ward, using the twistor formalism to reduce the problem to one of holomorphic vector bundles on the algebraic surface $T\mathbb{P}_1$, the tangent bundle of the projective line. Ward's method, extended by Corrigan and Goddard [6] and the author [8], shows that the monopole is determined by an algebraic curve in $T\mathbb{P}_1$. Moreover, as shown in [8], every monopole may be obtained in this way. The main problem of this approach is finding the conditions to impose on the curve in order to ensure that the monopole is non-singular.

The alternative approach, due to Nahm [10], incorporates the non-singularity condition directly and has other formal advantages over the twistor viewpoint. Nahm's method is a bold adaptation of the ADHM construction of instantons [3], replacing matrices by differential operators and the quadratic constraint on the matrices by a non-linear ordinary differential equation:

$$\frac{dT_i}{dz} = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}[T_j, T_k]$$